Review notes for exam this Friday November 13.

C.I.F. & applications

- Exam will cover 2.4-2.5, 3.1-3.3, 4.1-4.2, and implicitly use the earlier course material.

email me (korevaar "at" math.utah.edu) your preferred two hour time slot on Friday, starting on the hour between 10:00 a.m. and 4:00 p.m.

I will email you a .pdf of the exam at the start time or a few minutes early. Unless you tell me otherwise I'll email it to the return address of the email you send me.

Complete the exam and upload a .pdf of your solutions to Gradescope by two hours after your start time. For insurance or if you have trouble uploading, email me a .pdf within your time limit as well.

The exam is closed book, closed notes, closed internet etc. Your only resource is yourself. I'll ask you to sign an honor-code like statement on the front of your exam which will be part of what you upload to Gradescope.

As with the first exam there will be some required problems at the beginning of the exam, and then you'll be asked to complete several substantial problems where you have some choice about which problems to tackle. There will be a mixture of theorem proofs/explanations, along with computations.

There is a practice exam posted in our CANVAS notes, and I'll go over it on Thursday during office hours, starting at 2:00.

Topics:

2.4 Cauchy integral formula

possible exam things

Index $I(\gamma; z_0) = \frac{1}{2\pi}\int_a^b \Theta(t)\,dt = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-z_0}\,dz$ ($\Theta(t)$: $\arg(z-z_0)$)

* C.I.F. for closed contour γ contractible in a domain on which $f(z)$ is analytic. $f(z)\, I(\gamma; z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta$

formulas and estimates for derivatives

* $f'(z) I(\gamma; z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^2}\,d\zeta$

$(f^{(k)}(z) \ldots)$

* Liouville's Theorem : bd entire fcn is constant

* Fundamental Theorem of Algebra

* Morera's Theorem. cont f is analytic iff Rectangle lemma holds

key, e.g. uniform limits of analytic fcns are analytic.

2.5 Maximum modulus principle and harmonic functions no thm pfs in this section. know statements & be able to use

Mean value property for $f(z)$ analytic : $f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta$ was special

used

Clever proof for harmonic conjugates in simply connected domains not on exam

Mean value property for harmonic functions

Maximum modulus principle for $f(z)$ analytic

Maximum and minimum principles for harmonic functions

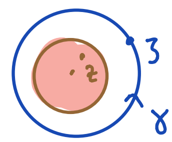
3.1 Convergent sequences and series of analytic functions

Via Morera's Thm

why uniform limits of analytic functions are analytic, and why the derivative of the limit of analytic functions is the limit of the derivatives

C.I.F.

$$f_n'(z) = \frac{1}{2\pi i}\int_\gamma \frac{f_n(\zeta)}{(\zeta - z)^2}\,d\zeta$$



↑ by unif conv.

$f_n'(z) \to f'(z)$

- Weierstrass M test

3.2 Power series and Taylor's Theorem

- radius of convergence
- term by term differentiation/integration R stays same
- uniqueness for given f because they're Taylor series
- analytic if and only if power series unif conv. use geometric series magic in C.I.F.

* isolated zeroes theorem using Taylor series @ z_0, $f(z_0)=0$ $f(z) = (z-z_0)^N g(z)$ via Taylor, $g(z_0) \neq 0$

- multiplication of power series

key examples $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ $|z|<1$, $\underbrace{e^z, \cos z, \sin z}_{@\ z_0=0}$, $(1+z)^\alpha$, integrals, derivs, $\log(1+z)$ ...

know fact & how to use

3.3 Laurent series

unif conv.

analytic in an annulus (including punctured disk case) if and only if Laurent series.... .where does each piece converge?

* uniqueness geometric series magic in C.I.F.

- isolated ~~zeroes~~ singularities classification
  - removable → L.S. no neg. powers
  - poles → finitely many neg powers in L.S.
  - essential. ∞ " " " "

residue @ z_0 $\mathrm{Res}(f; z_0) = $ coef of $\frac{1}{z-z_0}$ term.

- multiplication of Laurent series.
- geometric series wizardry.

"big" "little" magic using geometric series

favorite exam question

4.1 Calculating residues at isolated singularities

$$f(z) = \frac{f_1(z)}{(z-z_0)^k} + f_2(z)$$

$\uparrow$ analytic.

in e.g. hw.

$$f(z) = \frac{g(z)}{h(z)} = \frac{\sum_{n=M}^{\infty} a_n (z-z_0)^n}{\sum_{n=N}^{\infty} \tilde{a}_n (z-z_0)^n}$$

simple poles easiest case

table will be provided

$\{z_1, \ldots z_n\}$ isolated sings in A

4.2 Residue theorem

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{j=1}^{n} \mathrm{Res}(f; z_j)\, I(\gamma; z_j)$$

* statement and proof for γ contractible in A via the deformation theorem

* statement and proof if γ is a simple closed curve bounding a domain via Replacement Theorem for domains with holes.

- contour integral computations via the residue theorems
- residues at ∞ (I'll remind you of the formula if you have to use it.)
  - & Res Thm.

Math 4200
Wednesday November 11

4.1-4.2 homework discussion; exam review

Announcements:

The floor is open for discussion of the homework due Friday! Groups? In the ~~second~~ first portion of class we'll go over the review outline in Monday's notes.

There is a section 4.3 homework assignment due next week Friday -see page 2. It's applications of the Residue Theorem to computing definite integrals, many of which are not accessible using regular Calculus techniques.

all HW solt'ns are posted
I'll finish grading HW 7, 8 tonight
9, 10 before 2:00 tomorrow.

Th 2:00 office hours ~ go over practice test.

Math 4200-001
Homework 12
4.3
Due Friday November 20 at 11:59 p.m.

4.3: 1, 2, 4, 6, 10, 14, 17, 20ab.
There are a lot of good worked examples in the text. In problem 6 you may use entry #5 on the Definite integral table 4.3.1, page 296. The text explains why this table entry is true on pages 289-293 and summarizes it as Proposition 4.3.16. We'll also discuss 6, 14 some in class on Monday

Math 4200-001
Homework 11
4.1-4.2
Due Friday November 11 at 11:59 p.m.
Exam will cover thru 4.2

4.1 1de, 3, 5, 7ab, 9
4.2 2 (Section 2.3 Cauchy's Theorem), 3, 4, 6, 9, 13.

w11.1 (extra credit) Prove Prop 4.1.7, the determinant computation for the residue at an order k pole for $f(z) = \dfrac{g(z)}{h(z)}$ at z_0, where $g(z_0) \neq 0$. (Hint: it's Cramer's rule for a system of equations.)

4.1.1 Find residues:

d) $\dfrac{1 + e^z}{z^4}$, $z_0 = 0$; e) $\dfrac{e^z}{\left(z^2 - 1\right)^2}$, $z_0 = 1$

$\frac{1}{z^4}\left(g(z)\right)$

$g(z_0) \neq 0.$

this might merit the table for double pole.

$\frac{1}{z^4}\left[1 + 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right]$

$\frac{1}{z}$ coef is $\frac{1}{3!}$

4.1.3 Show by example that $Res\left(f(z)^2, z_0\right) \neq \left(Res\left(f(z), z_0\right)\right)^2$ in general.

4.1.5 What fails in this reasoning: Let

$$f(z) = \frac{1+e^z}{z^2} + \frac{1}{z}$$

Since $f(z)$ has a pole at $z = 0$ the residue of f at that point is the coefficient of $\frac{1}{z}$ there, namely 1.

4.1.7 Find all singular points and residues:

a) $\dfrac{1}{z^3(z+4)}$ b) $\dfrac{1}{z^2+z+1}$ c) (not assigned) $\dfrac{1}{z^3-3}$

4.1.9 Find the residue of $\dfrac{1}{z^2 \underbrace{\sin(z)}_{\left(z - \frac{z^3}{3!} + \cdots\right)}}$ at $z = 0$ Compute the series.

$\frac{1}{z^3}\left[\text{analytic} \atop g(z_0) \neq 0\right]$ pole order 3

$$\frac{1}{z^2 \sin z} = \left(\frac{c_{-3}}{z^3} + \frac{c_{-2}}{z^2} + \frac{c_{-1}}{z} + c_0 + \cdots \right)$$

$$1 = \left(\frac{c_{-3}}{z^3} + \frac{c_{-2}}{z^2} + \frac{c_{-1}}{z} + c_0 + \cdots \right) z^2 \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \right)$$

$$1 = \left(\frac{c_{-3}}{z} + c_{-2} + c_{-1} z + \cdots \right) \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \right).$$

z^0 : $1 = c_{-3}$

z^1 : $0 = c_{-2}$

z^2 : $0 = -\frac{1}{6} c_{-3} + c_{-1} \Rightarrow c_{-1} = \frac{1}{6}.$

4.2 2 (Section 2.3 Cauchy's Theorem), 3, 4, 6, 9, 13.

2) Deduce Cauchy Integral Formula from Residue Theorem

3) Evaluate $\int_\gamma \frac{z}{z^2 + 2z + 5} dz$ where γ is the unit circle.

4) Find $\int_\gamma \frac{1}{e^z - 1} dz$ where γ is the circle of radius 9 and center zero.

$f(z) = \frac{g(z)}{h(z)}$

$g(z_0) \neq 0$

$h(z_0) = 0, \quad h'(z_0) \neq 0$

where are the singularities?

$\mathrm{Res}(f, z_0) = \frac{g(z_0)}{h'(z_0)}$

$z = 0, 2\pi i, -2\pi i$

6) Show $\int_\gamma \frac{5z - 2}{z(z-1)} dz = 10\pi i$ where γ is any circle of center 0 and radius greater than 1.

power series tricks are often quicker than table.

Office hrs from 1-2 today also

9) Evaluate

a) $$\int_{|z|=\frac{1}{2}} \frac{dz}{z(1-z)^3}$$

b) $$\int_{|z|=\frac{1}{2}} \frac{e^z\, dz}{z(1-z)^3}$$

13a) Find $Res\left(\frac{(z-1)^3}{z(z+2)^3}; \infty\right)$ Recall, $Res(f;\ \infty) := Res\left(-\frac{1}{z^2} f\left(\frac{1}{z}\right); 0\right)$

13b) Compute $\int_{|z|=3} \frac{(z-1)^3}{z(z+2)^3}\, dz$ two ways.